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LETTER TO THE EDITOR

Equivalence of two character formulae for typical and singly atypical irreducible modules of sl(m/n)

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Abstract. The Bernstein–Leites–Van der Jeugt character formula, known to be valid for all typical and singly atypical irreducible modules of sl(m/n), is shown to be equivalent in all such cases to the formula conjectured by Hughes and King, which previously has only been proved valid in the case of sl($m/1$).

Interest in the representation theory of Lie superalgebras continues because of actual and potential applications in physics (see, for example, the collection of papers edited by Kostelecky and Campbell (1985)). Character formulae provide a concise and powerful means of specifying the weight content of finite-dimensional irreducible modules of such algebras. Kac (1978) showed how to generalise from Weyl's (1926) character formula for any finite-dimensional irreducible module of a semisimple Lie algebra, to the case of a so-called *typical* irreducible module of any of the basic classical superalgebras. Subsequent attempts to obtain formulae valid for the poorly understood *atypical* modules have been only partially successful. In the important case of the superalgebra sl(m/n), the situation as it stood in 1987 was summarised in a letter to the editor of this journal (Hughes and King 1987).

Bernstein and Leites (1980) proposed a character formula for all typical and atypical irreducible modules of this algebra, but it was later found that their formula is not valid in all atypical cases (Leites 1987, Hughes and King 1987).

An equivalent formula was described by Van der Jeugt (1987), who suggested a much more limited range of applicability, in particular to irreducible modules that are at most *singly* atypical (i.e. modules that are atypical with respect to at most one odd positive root). In fact, Leites (1987) has asserted that the Bernstein–Leites–Van der Jeugt (BLV) formula does remain valid for all finite-dimensional irreducible modules of sl($m/1$); these are all either typical or singly atypical.

Hughes and King (1987) conjectured the validity, for all typical and atypical irreducible sl(m/n) modules, of a different character formula, but it is now known that this too is not generally valid, and yet another formula has recently been proposed (Hughes *et al* 1989a).

In the midst of this uncertainty, it has now been proved that the BLV formula is indeed correct for all typical and singly atypical irreducible modules of sl(m/n) (Hughes *et al* 1989b), and also that the Hughes–King (HK) formula is correct for all finite-dimensional irreducible modules of sl($m/1$) (Gould *et al* 1989). It follows that the BLV and HK formulae are equivalent for sl($m/1$), although this is certainly not obvious on inspection.

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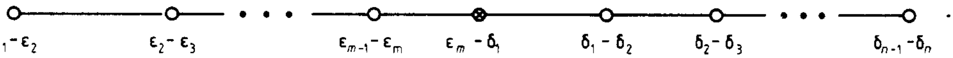
It is the purpose of this letter to prove that in fact the BLV and HK formulae are equivalent for all typical and singly atypical irreducible modules of $sl(m/n)$. Their equivalence for all finite-dimensional irreducible modules of $sl(m/1)$ is thereby explained, and it now follows moreover that the HK formula, like the BLV formula, is valid for all typical and singly atypical irreducible modules of $sl(m/n)$. The extent of any wider range of validity remains unknown for both formulae, but it seems clear at least that the HK formula is valid for some multiply atypical modules of $sl(m/n)$ where the BLV formula is invalid (Hughes and King 1987), so they are not always equivalent.

To describe the two character formulae, we need to establish some notation. The $sl(m/n)$ algebra is most easily characterised by the following Dynkin diagram, labelled by a choice of simple roots:



On the $(m+n)$-dimensional vector space spanned by the basis $\{\varepsilon_i, \delta_\mu\,;\, i=1,2,\ldots,m;\ \mu=1,2,\ldots,n\}$, an inner product deduced from the Killing form for the superalgebra is defined by

$$(\varepsilon_i, \varepsilon_j)=\delta_{ij} \qquad (\delta_\mu, \delta_\nu)=-\delta_{\mu\nu} \qquad (\varepsilon_i, \delta_\nu)=0 \qquad \forall i, j, \mu, \nu. \tag{1}$$

The set of even positive roots is determined from the diagram to be

$$\Phi_0^+=\{\varepsilon_i-\varepsilon_j,\ i<j;\ \delta_\mu-\delta_\nu,\ \mu<\nu\} \tag{2}$$

which is actually the union of two disjoint subsets:

$$\Phi_0^+(sl(m))=\{\varepsilon_i-\varepsilon_j,\ i<j\} \qquad \Phi_0^+(sl(n))=\{\delta_\mu-\delta_\nu,\ \mu<\nu\} \tag{3}$$

corresponding to the positive roots of $sl(m)$ and $sl(n)$ subalgebras, respectively. The set of odd positive roots is determined to be

$$\Phi_1^+=\{\varepsilon_i-\delta_\mu\,;\ i=1,2,\ldots,m,\ \mu=1,2,\ldots,n\}. \tag{4}$$

Denote the halved-sums of the even and odd positive roots respectively by

$$\rho_0=\frac{1}{2}\sum_{\alpha\in\Phi_0^+}\alpha \qquad \rho_1=\frac{1}{2}\sum_{\gamma\in\Phi_1^+}\gamma \tag{5}$$

and set

$$\rho=\rho_0-\rho_1. \tag{6}$$

Consider a finite-dimensional irreducible $sl(m/n)$ module $V(\Lambda)$ with highest weight

$$\Lambda=\sum_{i=1}^{m}\lambda_i\varepsilon_i+\sum_{\mu=1}^{n}\omega_\mu\delta_\mu=(\lambda_1,\lambda_2\ldots\lambda_m|\omega_1,\omega_2\ldots\omega_n). \tag{7}$$

Here λ_i, $\omega_\mu\in\mathbb{C}$, $\Sigma_{i=1}^{n}\lambda_i+\Sigma_{\mu=1}^{n}\omega_\mu=0$ and $\lambda_i-\lambda_{i+1}$ and $\omega_\mu-\omega_{\mu+1}$ are non-negative integers for $i=1,2,\ldots,m-1$, $\mu=1,2,\ldots,n-1$. We say that the module, the corresponding representation and the highest weight Λ are typical if the subset of odd positive roots

$$\Phi_a^+(\Lambda)=\{\gamma\in\Phi_1^+|(\Lambda+\rho,\gamma)=0\} \tag{8}$$

is empty, otherwise we say they are atypical. In particular, if the order of $\Phi_a^+(\Lambda)$ is 1, we say the module etc are singly atypical. Following Hughes and King (1987), we define also the subset of odd positive roots

$$\Phi_b^+(\Lambda)=\{\gamma\in\Phi_1^+\backslash\Phi_a^+(\Lambda)|\gamma=\gamma'+\alpha,\ \gamma'\in\Phi_a^+(\Lambda),\ \alpha\in\Phi_0^+,\ (\Lambda,\alpha)=0\}. \tag{9}$$

The two character formulae in question are then the following.

(i) The Bernstein–Leites–Van der Jeugt formula

$$ch_{\mathrm{BLV}}(\Lambda) = q^{-1} \sum_{\sigma \in W} \mathrm{sn}(\sigma) \exp[\sigma(\Lambda + \rho_0)] \prod_{\gamma \in \Phi_r^+(\Lambda)} (1 + \mathrm{e}^{-\sigma(\gamma)}). \tag{10}$$

(ii) The Hughes–King formula

$$ch_{\mathrm{HK}}(\Lambda) = q^{-1} \sum_{\sigma \in W} \mathrm{sn}(\sigma) \exp[\sigma(\Lambda + \rho_0)] \prod_{\gamma \in \Phi_s^+(\Lambda)} (1 + \mathrm{e}^{-\sigma(\gamma)}) \tag{11}$$

where W denotes the Weyl group of the even subalgebra $\mathrm{sl}(m) \oplus \mathrm{sl}(n)$, and

$$q = \prod_{\alpha \in \Phi_0^+} (\mathrm{e}^{\alpha/2} - \mathrm{e}^{-\alpha/2})$$
$$\Phi_r^+(\Lambda) = \Phi_1^+ \backslash \Phi_a^+(\Lambda) \qquad \Phi_s^+(\Lambda) = \Phi_1^+ \backslash (\Phi_a^+(\Lambda) \cup \Phi_b^+(\Lambda)). \tag{12}$$

Both formulae reduce to Kac's established formula (Kac 1978) in the case of typical modules, when $\Phi_a^+(\Lambda)$ and $\Phi_b^+(\Lambda)$ are empty. We now show that the two formulae (10) and (11) are also equivalent for singly atypical irreducible modules.

It is convenient to use the following symbolic notation to indicate an atypicality condition on a highest weight:

$$\Lambda = (\lambda_1, \lambda_2 \ldots \lambda_{l-1}, \overline{\lambda, \lambda_{l+1} \ldots \lambda_m | \omega_1, \omega_2 \ldots \omega_{\xi-1}, \omega}, \omega_{\xi+1} \ldots \omega_n) \tag{13}$$

corresponding to the condition

$$(\Lambda + \rho, \varepsilon_l - \delta_\xi) = 0 \tag{14}$$

or equivalently, because of (1), (6) and (7), to the condition

$$\lambda + \omega = l + \xi - m - 1. \tag{15}$$

For multiply atypical highest weights, more than one pair (λ_i, ω_μ) is 'contracted' as in (13), and it can easily be seen from (15) and the lexicality conditions on the λ_i and ω_μ, that the lines contracting different pairs can never intersect, but must be nested inside one another.

Restricting our attention to the singly atypical Λ of (13), we note that since we can write

$$ch_{\mathrm{BLV}}(\Lambda) = q^{-1} \sum_{\sigma \in W} \mathrm{sn}(\sigma) \exp[\sigma(\Lambda + \rho_0)] \prod_{\gamma' \in \Phi_b^+(\Lambda)} (1 + \mathrm{e}^{-\sigma(\gamma')}) \prod_{\gamma \in \Phi_s^+(\Lambda)} (1 + \mathrm{e}^{-\sigma(\gamma)}) \tag{16}$$

then $ch_{\mathrm{BLV}}(\Lambda)$ and $ch_{\mathrm{HK}}(\Lambda)$ are obviously equivalent if $\Phi_b^+(\Lambda)$ is empty. Therefore we only need to consider the case that $\Phi_b^+(\Lambda)$ is non-empty. In the case under discussion

$$\Phi_a^+(\Lambda) = \{\varepsilon_l - \delta_\xi\} \qquad |\Phi_a^+(\Lambda)| = 1 \tag{17}$$

so that, for any $\gamma \in \Phi_b^+(\Lambda)$, we have

$$\gamma = (\varepsilon_l - \delta_\xi) + \alpha \qquad (\Lambda, \alpha) = 0 \tag{18}$$

with

$$\alpha \in \Phi_0^+(\mathrm{sl}(m)) \qquad \text{or} \qquad \alpha \in \Phi_0^+(\mathrm{sl}(n)). \tag{19}$$

Suppose there exist certain $\alpha_i \in \Phi_0^+(\mathrm{sl}(m))$ such that (18) is satisfied. Then each such α_i must have the form

$$\alpha_i = \varepsilon_i - \varepsilon_l \qquad \text{for some } i < l \tag{20}$$

and the second of equations (18) requires that Λ have the form

$$\Lambda_1 = (\lambda_1, \lambda_2 \ldots \lambda_k, \lambda, \lambda \ldots \overline{\lambda, \lambda_{l+1} \ldots \lambda_m | \omega_1, \omega_2 \ldots \omega_{\xi-1}, \omega}, \omega_{\xi+1} \ldots \omega_n)$$
$$\lambda_k > \lambda \qquad 1 \leq k < l \qquad \omega_{\xi+1} < \omega. \tag{21}$$

The condition $\omega_{\xi+1}<\omega$ follows from the single atypicality of Λ_1, since if $\omega_{\xi+1}$ were equal to ω, the root $\varepsilon_{l-1}-\delta_{\xi+1}$ would also be atypical (cf (15)), and Λ_1 would be at least doubly atypical. The assumption of single atypicality also rules out the existence of any $\alpha\in\Phi_0^+(\mathrm{sl}(n))$ satisfying (18) when $\Lambda=\Lambda_1$. Thus

$$\Phi_h^+(\Lambda_1)=\{\varepsilon_i-\delta_\xi|i=k+1,k+2,\ldots,l-1\}. \tag{22}$$

Similarly, if we assume the existence of some $\alpha_\mu\in\Phi_0^+(\mathrm{sl}(n))$ satisfying (18), we will have

$$\begin{aligned}&\Lambda_2=(\lambda_1,\lambda_2\ldots\lambda_{l-1},\overline{\lambda,\lambda_{l+1}\ldots\lambda_m|\omega_1,\omega_2\ldots\omega_{\xi-1},\omega},\omega\ldots\omega,\omega_\tau\ldots\omega_n)\\&\omega_\tau<\omega\qquad\xi<\tau\leqslant n\qquad\lambda_{l-1}>\lambda\end{aligned} \tag{23}$$

and

$$\Phi_h^+(\Lambda_2)=\{\varepsilon_l-\delta_\mu|\mu=\xi+1,\xi+2,\ldots,\tau-1\}. \tag{24}$$

We emphasise that Λ_1 and Λ_2 are the only two types of singly atypical highest weights with non-empty $\Phi_h^+(\Lambda)$.

Consider Λ_1 first. Using (16), expanding the product over $\Phi_h^+(\Lambda_1)$ and combining term by term into the leftmost exponential, we arrive at

$$\begin{aligned}&ch_{\mathrm{BLV}}(\Lambda_1)=q^{-1}\sum_{\Lambda_1'}S(\Lambda_1')\\&S(\Lambda_1')=\sum_{\sigma\in W}\mathrm{sn}(\sigma)\exp[\sigma(\Lambda_1'+\rho_0)]\prod_{\gamma\in\Phi_s^+(\Lambda_1)}(1+\mathrm{e}^{-\sigma(\gamma)})\end{aligned} \tag{25}$$

where Λ_1' runs over all weights of the form

$$\begin{aligned}\Lambda_1'=(&\lambda_1,\lambda_2\ldots\lambda_k,\lambda'_{k+1},\lambda'_{k+2}\ldots\lambda'_{l-1},\lambda,\lambda_{l+1}\ldots\lambda_m|\omega_1,\omega_2\\&\ldots\omega_{\xi-1},\omega',\omega_{\xi+1}\ldots\omega_n)\end{aligned} \tag{26}$$

with

$$\begin{aligned}&\lambda_i'=\lambda\qquad\text{or}\qquad\lambda-1\qquad i=k+1,k+2,\ldots,l-1\\&\omega'=\omega+\sum_{i=k+1}^{l-1}(\lambda-\lambda_i').\end{aligned} \tag{27}$$

For a given $\Lambda_1'\neq\Lambda_1$, there exists a t, $k<t<l$, such that

$$\lambda'_{t+1}=\lambda\qquad\lambda'_t=\lambda-1\qquad(\lambda'_l=\lambda\text{ understood}) \tag{28}$$

and under σ_t, the Weyl reflection with respect to the simple root $\varepsilon_t-\varepsilon_{t+1}$, we have

$$\sigma_t(\Lambda_1'+\rho_0)=\Lambda_1'+\rho_0\qquad\sigma_t(\Phi_s^+(\Lambda_1))=\Phi_s^+(\Lambda_1). \tag{29}$$

The first equation in (29) is obvious on noting that $(\rho_0,\varepsilon_t-\varepsilon_{t+1})=1$; to see the second one, we decompose $\Phi_s^+(\Lambda_1)$ into disjoint subsets

$$\Phi_s^+(\Lambda_1)=\left(\bigcup_{\mu\neq\xi}\Delta_\mu^+(\Lambda_1)\right)\cup\{\varepsilon_i-\delta_\xi|i=1,2,\ldots,k;\,l+1,l+2,\ldots,m\} \tag{30}$$

where

$$\Delta_\mu^+(\Lambda_1)=\{\varepsilon_i-\delta_\mu|i=1,2,\ldots,m\}. \tag{31}$$

Now it is apparent that all the subsets on the RHS of (30) are invariant under σ_t, so that $\Phi_s^+(\Lambda_1)$ itself is invariant. We can therefore rewrite

$$S(\Lambda_1') = \sum_{\sigma\in W} \mathrm{sn}(\sigma\sigma_t) \exp[\sigma\sigma_t(\Lambda_1'+\rho_0)] \prod_{\gamma\in\Phi_s^+(\Lambda_1)} \{1+\exp[-\sigma\sigma_t(\gamma)]\} \qquad \Lambda_1'\neq\Lambda_1 \quad (32)$$

which, with (25), immediately leads to

$$S(\Lambda_1') = -S(\Lambda_1')$$

i.e.

$$S(\Lambda_1') = 0 \qquad \forall\Lambda_1'\neq\Lambda_1. \quad (33)$$

Therefore

$$ch_{\mathrm{BLV}}(\Lambda_1) = q^{-1}S(\Lambda_1) = ch_{\mathrm{HK}}(\Lambda_1). \quad (34)$$

It can be shown in the same way that

$$ch_{\mathrm{BLV}}(\Lambda_1) = ch_{\mathrm{HK}}(\Lambda_2)$$

and we conclude that for typical and singly atypical irreducible modules of $\mathrm{sl}(m/n)$, the BLV and HK character formulae are indeed equivalent.

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